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Expansions in Laguerre Polynomials of Negative Order

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We discuss pointwise convergence for expansions of Laguerre polynomials of order $\alpha \leqslant -1$.

1. INTRODUCTION

For $\alpha > -1$, the Laguerre polynomials $\{L_m^\alpha(x)\}$ are defined by orthogonality:

$$\int_0^\infty L_k^\alpha(x)\, L_m^\alpha(x)\, e^{-x} x^\alpha\, dx = \frac{\Gamma(k+\alpha+1)}{k!}\delta_{k,m}, \qquad k, m = 0, 1, ..., \tag{1.1}$$

and the condition that $L_m^\alpha(x)$ is a polynomial of degree m with coefficient of x^m equal to $(-1)^m/m!$.

They are given explicitly by the formula:

$$L_m^\alpha(x) = \sum_{j=0}^m \frac{\Gamma(m+\alpha+1)}{\Gamma(j+\alpha+1)} \frac{(-1)^j x^j}{j!(m-j)!}, \qquad m = 0, 1, ..., \tag{1.2}$$

which extends the definition of Laguerre polynomials to all $\alpha \in C$. For a summary of the elementary properties of the Laguerre polynomials, see [4].

Denote by "*f.p.*" Hadamard's finite part of an infinite integral, defined below. It is known for non-integer $\alpha < -1$ that [3]:

$$f.p. \int_0^\infty L_k^\alpha(x)\, L_m^\alpha(x)\, e^{-x} x^\alpha\, dx = \frac{\Gamma(k+\alpha+1)}{k!}\delta_{k,m}, \qquad k, m = 0, 1, ..., \tag{1.3}$$

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Thus Laguerre expansions for non-integer $\alpha < -1$ may be defined by

$$f(x) \sim \sum_{k=0}^{\infty} a_k L_k^\alpha(x) \tag{1.4}$$

where

$$a_k = \frac{k!}{\Gamma(k+\alpha+1)} \text{f.p.} \int_0^\infty f(y)\, L_k^\alpha(y)\, e^{-y} y^\alpha\, dy. \tag{1.5}$$

In this paper we prove pointwise convergence of expansions of functions for which the difference of the function and a suitable polynomial satisfies a certain integrability condition. A simple example of such a function is e^{-cx} for $c > -1/2$ which is known to have the expansion

$$e^{-cx} = (1+c)^{-\alpha-1} \sum_{j=0}^{\infty} \left(\frac{c}{1+c}\right)^j L_j^\alpha(x)$$

which converges pointwise for all $x \in R$ and all complex α.

Denoting by $K_m^\alpha(x, y)$ the kernel polynomial

$$K_m^\alpha(x, y) = \sum_{j=0}^{m} \frac{j!}{\Gamma(j+\alpha+1)} L_j^\alpha(x)\, L_j^\alpha(y) \tag{1.6}$$

we have

$$\sum_{k=0}^{m} a_k L_k^\alpha(x) = \text{f.p.} \int_0^\infty f(y)\, K_m^\alpha(x, y)\, e^{-y} y^\alpha\, dy. \tag{1.7}$$

We will show for all non-integer $\alpha < -1$ that

$$\text{f.p.} \int_0^\infty K_m^\alpha(x, y)\, e^{-y} y^\alpha\, dy = 1 \tag{1.8}$$

and for all $\alpha \in R$ that

$$\lim_{m \to \infty} \int_a^b K_m^\alpha(x, y)\, e^{-y} y^\alpha\, dy = 1, \qquad 0 < a < x < b < \infty. \tag{1.9}$$

We use (1.3)–(1.9) to prove two theorems about pointwise convergence. First we show for $\alpha \leqslant -1/2$ that if

$$\int_0^1 |f(y)|\, y^\alpha\, dy < \infty \tag{1.10}$$

$$\int_1^\infty |f(y)|\, e^{-y/2} y^{((\alpha/2)+(5/12))}\, dy < \infty \tag{1.11}$$

and for $x \in (0, \infty)$ if

$$\frac{f(y)-f(x)}{y-x}$$

is locally integrable on $(0, \infty)$ as a function of y then

$$\lim_{m \to \infty} \int_0^\infty K_m^\alpha(x, y) f(y) e^{-y} y^\alpha \, dy = f(x). \tag{1.12}$$

This theorem enables us to define Laguerre expansions also for $\alpha = -1, -2, \ldots$ for certain functions.

Secondly, we show for $n = 0, 1, \ldots, -(n+2) < \alpha < -(n+1)$ that if there exists a polynomial $P_n(x)$ of degree n such that

$$\int_0^1 |f(y) - P_n(y)| \, y^\alpha \, dy < \infty, \tag{1.13}$$

if (1.11) holds, and for $x \in (0, \infty)$, if

$$\frac{f(y)-f(x)}{y-x}$$

is locally integrable on $(0, \infty)$ as a function of y then

$$\lim_{m \to \infty} f.p. \int_0^\infty K_m^\alpha(x, y) f(y) e^{-y} y^\alpha \, dy = f(x). \tag{1.14}$$

2. Generalized Orthogonality

Definition 2.1. *Let n be a non-negative integer, $-(n+2) < \alpha < -(n+1)$. Let $f(x)$ be a given function, and suppose there exists a polynomial $P_n(x)$ of degree n so that*

$$\int_0^\infty |f(x) - P_n(x)| \, x^\alpha \, dx < \infty. \tag{2.1}$$

Then Hadamard's finite part (f.p.) of the integral

$$\int_0^\infty f(x) \, x^\alpha \, dx$$

is defined by

$$f.p. \int_0^\infty f(x) \, x^\alpha \, dx = \int_0^\infty (f(x) - P_n(x)) \, x^\alpha \, dx. \tag{2.2}$$

For more information about this integral see [1].

Note that if $f(x), f'(x), ..., f^{(n+1)}(x)$ are defined on $[0, a]$ for some $a > 0$, if $f^{(n+1)}(x) \in L((0, a); x^{n+1+\alpha}\, dx)$ and $f(x) \in L((a, \infty); x^{\alpha}\, dx)$ then Hadamard's integral will be well-defined provided we take as $P_n(x)$ the n^{th} Taylor polynomial of $f(x)$ centered at 0.

THEOREM 2.2 [3]. *Let $\alpha < -1$ be non-integral. Then*

$$f.p. \int_0^{\infty} L_k^{\alpha}(x)\, L_m^{\alpha}(x)\, e^{-x} x^{\alpha}\, dx = \frac{\Gamma(k+\alpha+1)}{k!}\, \delta_{k,m},$$

$$k, m = 0, 1, ... \tag{2.3}$$

Orthogonality holds in a restricted sense for $\alpha = -l$, $l = 1, 2, ...$ provided $k, m \geqslant l$. Since for all $k \geqslant l$ [4]:

$$L_k^{(-l)}(x) = (-x)^l \frac{(k-l)!}{k!} L_{k-l}^{(l)}(x)$$

we have for $k, m \geqslant l$:

$$\begin{aligned} &\int_0^{\infty} L_k^{(-l)}(x)\, L_m^{(-l)}(x)\, e^{-x} x^{-l}\, dx \\ &\quad = \frac{(k-l)!\,(m-l)!}{k!m!} \int_0^{\infty} L_{k-l}^{(l)}(x)\, L_{m-l}^{(l)}(x)\, e^{-x} x^{l}\, dx \\ &\quad = \frac{(k-l)!}{k!}\, \delta_{k,m}. \end{aligned}$$

3. EXPANSIONS OF LAGUERRE POLYNOMIALS

Suppose for some nonintegral $\alpha < -1$ we have

$$f(x) = \sum_{j=0}^{\infty} a_j L_j^{\alpha}(x).$$

Then, assuming that the required integrals exist and that we can interchange summation and integration we have

$$\begin{aligned} f.p. \int_0^{\infty} f(x)\, L_k^{\alpha}(x)\, e^{-x} x^{\alpha}\, dx &= \sum_{j=0}^{\infty} a_j\, f.p. \int_0^{\infty} L_j^{\alpha}(x)\, L_k^{\alpha}(x)\, e^{-x} x^{\alpha}\, dx \\ &= a_k \frac{\Gamma(k+\alpha+1)}{k!} \end{aligned}$$

so that

$$a_k = \frac{k!}{\Gamma(k+\alpha+1)} f.p. \int_0^\infty f(x)\, L_k^\alpha(x)\, e^{-x} x^\alpha\, dx, \qquad k=0, 1, \dots \tag{3.1}$$

In particular, if $f(x)$ is a polynomial then the coefficients of its Laguerre expansion are given by (3.1).

On the other hand, suppose for a given function $f(x)$ that the integrals in (3.1) all exist. Denote by $S_m(x)$ the m^{th} partial sum

$$S_m(x) = \sum_{j=0}^{m} a_j L_j^\alpha(x). \tag{3.2}$$

Then

$$\begin{aligned} S_m(x) &= \sum_{j=0}^{m} \left(\frac{j!}{\Gamma(j+\alpha+1)} f.p. \int_0^\infty f(y)\, L_j^\alpha(y)\, e^{-y} y^\alpha\, dy \right) L_j^\alpha(x) \\ &= f.p. \int_0^\infty f(y)\, K_m^\alpha(x, y)\, e^{-y} y^\alpha\, dy \end{aligned}$$

where for all $\alpha \in R$

$$\begin{aligned} K_m^\alpha(x, y) &= \sum_{j=0}^{m} \frac{j!}{\Gamma(j+\alpha+1)} L_j^\alpha(x)\, L_j^\alpha(y) \\ &= \frac{(m+1)!}{\Gamma(m+\alpha+1)} \frac{L_m^\alpha(x)\, L_{m+1}^\alpha(y) - L_{m+1}^\alpha(x)\, L_m^\alpha(y)}{x-y} \\ &= \frac{(m+1)!}{\Gamma(m+\alpha+1)} \frac{L_{m+1}^\alpha(x)\, L_{m+1}^{\alpha-1}(y) - L_{m+1}^\alpha(y)\, L_{m+1}^{\alpha-1}(x)}{x-y}, \end{aligned} \tag{3.3}$$

see [4], p. 266. If $f(x)$ is a polynomial of degree k then

$$f(x) = f.p. \int_0^\infty f(y)\, K_m^\alpha(x, y)\, e^{-y} y^\alpha\, dy, \qquad m=k, k+1, \dots \tag{3.4}$$

In particular by taking $f(x) = 1$ we obtain

$$f.p. \int_0^\infty K_m^\alpha(x, y)\, e^{-y} y^\alpha\, dy = 1, \qquad m=0, 1, \dots \tag{3.5}$$

The situation for $\alpha = -l$, $l = 1, 2, \dots$, is similar. If

$$f(x) = \sum_{j=l}^{\infty} a_j L_j^{(-l)}(x)$$

then formally for $k \geqslant l$ we have

$$\int_0^\infty f(x)\, L_k^{(-l)}(x)\, e^{-x} x^{-l}\, dx = \sum_{j=l}^{\infty} a_j \int_0^\infty L_j^{(-l)}(x)\, L_k^{(-l)}(x)\, e^{-x} x^{-l}\, dx$$
$$= a_k \frac{\Gamma(k-l+1)}{k!}.$$

Furthermore, if

$$S_m(x) = \sum_{j=1}^{m} a_j L_j^{(-l)}(x), \qquad m \geqslant l$$

then

$$S_m(x) = \int_0^\infty f(y)\, K_m^{(-l)}(x, y)\, e^{-y} y^{-l}\, dy, \qquad m \geqslant l.$$

THEOREM 3.1. *Let $0 < a < x < b < \infty$. Then for all $\alpha \in R$ we have*

$$\lim_{m \to \infty} \int_a^b K_m^\alpha(x, y)\, e^{-y} y^\alpha\, dy = 1. \tag{3.6}$$

For the proof of Theorem 3.1, we will need the following intermediate results.

LEMMA 3.2 [5]. *Let $[a, b] \subset (0, \infty)$ and fix $x \in (a, b)$. If $R_m(x, y)$, $a \leqslant y \leqslant b$, $m = 1, 2, \ldots$, satisfy $|R_m(x, y)| \leqslant C$ and $|\frac{d}{dy} R_m(x, y)| \leqslant C m^{1/2}$ for some constant C independent of y and m, if $R_m(x, y)$ have continuous derivatives in y, and if $R_m(x, y)$ vanish for $y = x$ we have*

$$\lim_{m \to \infty} \frac{1}{m^{1/2}} \int_a^b \frac{R_m(x, y)}{x - y}\, dy = 0. \tag{3.7}$$

LEMMA 3.3 (Fejer's Formula, [4, p. 198]). *For all $\alpha \in R$ and $x > 0$*

$$L_m^\alpha(x) = \pi^{(-1/2)} m^{((\alpha/2) - (1/4))} e^{x/2} x^{(-(\alpha/2) - (1/4))}$$
$$\times \left(\cos\left\{ 2(mx)^{1/2} - \frac{2\alpha + 1}{4} \pi \right\} + \frac{\theta_{m,\alpha}(x)}{m^{1/2}} \right) \tag{3.8}$$

where $\theta_{m,\alpha}(x)$ is uniformly bounded for $x \in [a, b]$, $0 < a < b < \infty$, as $m \to \infty$.

LEMMA 3.4. *For any fixed a and b, $0<a<b<\infty$, there exists a constant C independent of x and m such that*

$$\left|\frac{d}{dx}\theta_{m,\alpha}(x)\right| \leqslant Cm^{1/2}, \qquad a\leqslant x\leqslant b. \tag{3.9}$$

Proof. Write $f_\alpha(x)=\pi^{-1/2}e^{x/2}x^{(-(\alpha/2)-(1/4))}$ and $\rho_{m,\alpha}(x)=\cos\{2(mx)^{1/2}-\frac{2\alpha+1}{4}\pi\}$. Note that $f_{\alpha+1}=x^{-1/2}f_\alpha$ and $\rho'_{m,\alpha}=-m^{1/2}\rho_{m,\alpha+1}/x^{1/2}$. Differentiating Fejer's formula we obtain

$$\begin{aligned}
&\frac{d}{dx}L_m^\alpha(x)\\
&=m^{((\alpha/2)-(1/4))}\left[f'_\alpha(x)\left(\rho_{m,\alpha}(x)+\frac{\theta_{m,\alpha}(x)}{m^{1/2}}\right)-f_\alpha(x)\right.\\
&\quad\left.\times\left(\frac{m^{1/2}}{x^{1/2}}\rho_{m,\alpha+1}(x)-\frac{\theta'_{m,\alpha}(x)}{m^{1/2}}\right)\right]\\
&=m^{((\alpha/2)-(1/4))}f'_\alpha(x)\left(\rho_{m,\alpha}(x)+\frac{\theta_{m,\alpha}(x)}{m^{1/2}}\right)\\
&\quad-m^{((\alpha/2)+(1/4))}f_{\alpha+1}(x)\,\rho_{m,\alpha+1}(x)\\
&\quad+m^{((\alpha/2)-(1/4))}f_\alpha(x)\frac{\theta'_{m,\alpha}(x)}{m^{1/2}}\\
&=m^{((\alpha/2)-(1/4))}f'_\alpha(x)\left(\rho_{m,\alpha}(x)+\frac{\theta_{m,\alpha}(x)}{m^{1/2}}\right)\\
&\quad-\left(L_m^{\alpha+1}(x)-m^{((\alpha/2)+(1/4))}f_{\alpha+1}(x)\frac{\theta_{m,\alpha+1}(x)}{m^{1/2}}\right)\\
&\quad+m^{((\alpha/2)-(1/4))}f_\alpha(x)\frac{\theta'_{m,\alpha}(x)}{m^{1/2}}.
\end{aligned}$$

Thus since [4]:

$$\frac{d}{dx}L_m^\alpha(x)=-L_{m-1}^{\alpha+1}(x) \qquad \text{and} \qquad L_m^\alpha(x)=L_m^{\alpha+1}(x)-L_{m-1}^{\alpha+1}(x)$$

we have

$$\begin{aligned}
L_m^\alpha(x)=m^{((\alpha/2)-(1/4))}&\left[f'_\alpha(x)\left(\rho_{m,\alpha}(x)+\frac{\theta_{m,\alpha}(x)}{m^{1/2}}\right)\right.\\
&\left.+f_{\alpha+1}(x)\,\theta_{m,\alpha+1}(x)+f_\alpha(x)\frac{\theta'_{m,\alpha}(x)}{m^{1/2}}\right].
\end{aligned}$$

Since $L_m^\alpha(x)=O(m^{((\alpha/2)-(1/4))})$ uniformly in $[a,b]$ and since $\rho_{m,\alpha}(x)$, $\theta_{m,\alpha}(x)$ and $\theta_{m,\alpha+1}(x)$ are uniformly bounded for all m and $x\in[a,b]$, this proves the lemma.

Proof of Theorem 3.1. Let

$$B_m=\frac{(m+1)!}{\Gamma(m+\alpha+1)}(m+1)^{\alpha-1}.$$

By Stirling's formula $\lim_{m\to\infty} B_m=1$.
Substituting Fejer's formula into

$$K_m^\alpha(x,y)=\frac{(m+1)!}{\Gamma(m+\alpha+1)}\frac{L_{m+1}^\alpha(x)\,L_{m+1}^{\alpha-1}(y)-L_{m+1}^\alpha(y)\,L_{m+1}^{\alpha-1}(x)}{x-y}$$

we obtain

$$K_m^\alpha(x,y)=B_m\frac{e^{(x+y)/2}(xy)^{(-(\alpha/2)-(1/4))}}{\pi(x-y)}\left(T_{m+1}(x,y)+\frac{U_{m+1}(x,y)}{(m+1)^{1/2}}\right) \tag{3.10}$$

where

$$T_m(x,y)=y^{1/2}\cos\left\{2(mx)^{1/2}-\frac{2\alpha+1}{4}\pi\right\}\sin\left\{2(my)^{1/2}-\frac{2\alpha+1}{4}\pi\right\}$$
$$-x^{1/2}\cos\left\{2(my)^{1/2}-\frac{2\alpha+1}{4}\pi\right\}\sin\left\{2(mx)^{1/2}-\frac{2\alpha+1}{4}\pi\right\}$$

and

$$U_m(x,y)=\theta_{m,\alpha}(x)\,y^{1/2}\left(\sin\left\{2(my)^{1/2}-\frac{2\alpha+1}{4}\pi\right\}+\frac{\theta_{m,\alpha-1}(y)}{m^{1/2}}\right)$$
$$-\theta_{m,\alpha}(y)\,x^{1/2}\left(\sin\left\{2(mx)^{1/2}-\frac{2\alpha+1}{4}\pi\right\}+\frac{\theta_{m,\alpha-1}(x)}{m^{1/2}}\right)$$
$$+y^{1/2}\theta_{m,\alpha-1}(y)\cos\left\{2(mx)^{1/2}-\frac{2\alpha+1}{4}\pi\right\}$$
$$-x^{1/2}\theta_{m,\alpha-1}(x)\cos\left\{2(my)^{1/2}-\frac{2\alpha+1}{4}\pi\right\}.$$

Note that $U_m(x, y)\, e^{-y/2}\, y^{((\alpha/2)-(1/4))}$ satisfies the conditions of Lemma 3.2, so that

$$\lim_{m\to\infty} \int_a^b K_m^\alpha(x, y)\, e^{-y} y^\alpha\, dy = \frac{e^{x/2} x^{(-(\alpha/2)-(1/4))}}{\pi} \times \lim_{m\to\infty} \left(\int_a^b \frac{T_m(x, y)}{x-y} e^{-y/2} y^{((\alpha/2)-(1/4))}\, dy + \frac{1}{m^{1/2}} \int_a^b \frac{U_m(x, y)}{x-y} e^{-y/2} y^{((\alpha/2)-(1/4))}\, dy \right)$$
$$= \frac{e^{x/2} x^{(-(\alpha/2)-(1/4))}}{\pi} \times \lim_{m\to\infty} \int_a^b \frac{T_m(x, y)}{x-y} e^{-y/2} y^{((\alpha/2)-(1/4))}\, dy.$$

Let us write

$$\frac{T_m(x, y)}{x-y} = \frac{-\cos\{2(mx)^{1/2} - \frac{2\alpha+1}{4}\pi\} \sin\{2(my)^{1/2} - \frac{2\alpha+1}{4}\pi\}}{y^{1/2} + x^{1/2}} + x^{1/2} \frac{\sin(2m^{1/2}(x^{1/2} - y^{1/2}))}{x-y}$$
$$= T_{1,m}(x, y) + x^{1/2} T_{2,m}(x, y).$$

By the Riemann–Lebesgue lemma,

$$\lim_{m\to\infty} \int_a^b T_{1,m}(x, y)\, e^{-y/2} y^{((\alpha/2)-(1/4))}\, dy = 0.$$

Let $\phi(y) = e^{-y/2} y^{((\alpha/2)-(1/4))}$, fix $\delta > 0$ so that $(x-\delta, x+\delta) \subset (a, b)$ and fix $\eta > 0$ such that $(-\eta, \eta) \subset (\sqrt{x-\delta} - \sqrt{x}, \sqrt{x+\delta} - \sqrt{x})$. Again using the Riemann–Lebesgue lemma we have

$$\lim_{m\to\infty} \int_a^b T_{2,m}(x, y)\, \phi(y)\, dy$$
$$= \lim_{m\to\infty} \int_{x-\delta}^{x+\delta} \frac{\sin(2\sqrt{m}(\sqrt{x} - \sqrt{y}))}{x-y} \phi(y)\, dy$$
$$= \lim_{m\to\infty} \int_{\sqrt{x-\delta}-\sqrt{x}}^{\sqrt{x+\delta}-\sqrt{x}} \frac{\sin(2\sqrt{m}\, v)}{v(v+2\sqrt{x})} (2v + 2\sqrt{x})\, \phi((v+\sqrt{x})^2)\, dv$$

$$= \lim_{m\to\infty}\left(\int_{\sqrt{x-\delta}-\sqrt{x}}^{\sqrt{x+\delta}-\sqrt{x}} \frac{\sin(2\sqrt{m}\,v)}{(v+2\sqrt{x})}\,\phi((v+\sqrt{x})^2)\,dv\right.$$
$$\left.+\int_{\sqrt{x-\delta}-\sqrt{x}}^{\sqrt{x+\delta}-\sqrt{x}} \frac{\sin(2\sqrt{m}\,v)}{v}\,\phi((v+\sqrt{x})^2)\,dv\right)$$
$$= \lim_{m\to\infty}\int_{-\eta}^{\eta} \frac{\sin(2\sqrt{m}\,v)}{v}\,\phi((v+\sqrt{x})^2)\,dv.$$

Note that since

$$\lim_{m\to\infty}\int_{-\eta}^{\eta}\frac{\sin(2\sqrt{m}\,v)}{v}\,dv = \lim_{m\to\infty}\int_{-2\eta\sqrt{m}}^{2\eta\sqrt{m}}\frac{\sin u}{u}\,du = \pi$$

we have

$$\lim_{m\to\infty}\int_{-\eta}^{\eta}\frac{\sin(2\sqrt{m}\,v)}{v}\,\phi((v+\sqrt{x})^2)\,dv - \pi\phi(x)$$
$$= \lim_{m\to\infty}\int_{-\eta}^{\eta}\sin(2\sqrt{m}\,v)\,\frac{\phi((v+\sqrt{x})^2)-\phi(x)}{v}\,dv = 0.$$

Therefore

$$\lim_{m\to\infty}\int_a^b T_{2,m}(x,y)\,e^{-y/2}y^{((\alpha/2)-(1/4))}\,dy = \pi e^{-x/2}x^{((\alpha/2)-(1/4))}.$$

This completes the proof of the theorem.

THEOREM 3.5. *Let $\alpha \leqslant -1/2$. Suppose*

$$\int_0^1 |f(y)|\,y^\alpha\,dy < \infty \tag{3.11}$$

$$\int_1^\infty |f(y)|\,e^{-y/2}y^{((\alpha/2)+(5/12))}\,dy < \infty. \tag{3.12}$$

Then for all points $x \in (0,\infty)$ for which the function

$$\frac{f(y)-f(x)}{y-x} \tag{3.13}$$

is locally integrable on $(0,\infty)$ as a function of y we have

$$\lim_{m\to\infty}\int_0^\infty K_m^\alpha(x,y)\,f(y)\,e^{-y}y^\alpha\,dy = f(x). \tag{3.14}$$

Proof. First observe that for any $[a, b] \subset (0, \infty)$ and $a < x < b$ we have

$$\begin{aligned}
\limsup_{m\to\infty} &\left| \int_0^\infty f(y)\, K_m^\alpha(x, y)\, e^{-y} y^\alpha\, dy - f(x) \right| \\
&= \limsup_{m\to\infty} \left| \int_0^\infty f(y)\, K_m^\alpha(x, y)\, e^{-y} y^\alpha\, dy - f(x) \int_a^b K_m^\alpha(x, y)\, e^{-y} y^\alpha\, dy \right| \\
&= \limsup_{m\to\infty} \left| \int_0^a f(y)\, K_m^\alpha(x, y)\, e^{-y} y^\alpha\, dy \right. \\
&\quad + \int_a^b (f(y) - f(x))\, K_m^\alpha(x, y)\, e^{-y} y^\alpha\, dy \\
&\quad \left. + \int_b^\infty f(y)\, K_m^\alpha(x, y)\, e^{-y} y^\alpha\, dy \right| \\
&= \limsup_{m\to\infty} |I_{m,1} + I_{m,2} + I_{m,3}|.
\end{aligned}$$

We will show that for any $\varepsilon > 0$, there exist numbers a and b, $0 < a < b < \infty$ so that

$$|I_{m,1}| + |I_{m,3}| < \varepsilon$$

uniformly in m. We will also show for any a and b, $0 < a < b < \infty$, that

$$\lim_{m\to\infty} I_{m,2} = 0.$$

Hence, we will have

$$\limsup_{m\to\infty} \left| \int_0^\infty f(y)\, K_m^\alpha(x, y)\, e^{-y} y^\alpha\, dy - f(x) \right| < \varepsilon.$$

Since ε can be chosen arbitrarily small, this will prove the theorem.

We first consider $I_{m,1}$. For any fixed number w, if $0 \leqslant x \leqslant w$, $\alpha \leqslant -1/2$, we have ([4], p. 178)

$$L_m^\alpha(x) = O(m^{((\alpha/2)-(1/4))}), \qquad m \to \infty,$$

where the bound is uniform in x. Therefore for $0 \leqslant y \leqslant a$ and fixed $x > a$ we have

$$\begin{aligned}
K_m^\alpha(x, y) = O(m^{1-\alpha})(&m^{((\alpha/2)-(1/4))} m^{((\alpha-1)/2)-(1/4)} \\
&+ m^{((\alpha-1)/2)-(1/4)} m^{((\alpha/2)-(1/4))}) = O(1)
\end{aligned}$$

uniformly in y. Therefore

$$|I_{m,1}| \leqslant \int_0^a |f(y)|\, K_m^\alpha(x, y)\, e^{-y} y^\alpha\, dy = O(1) \int_0^a |f(y)|\, e^{-y} y^\alpha\, dy$$

which can be made arbitrarily small independently of m by taking a sufficiently small.

Next, we consider $I_{m,3}$. For $\alpha \in R$, and for any fixed number $c > 0$ and all $x \geqslant c$, we have

$$L_m^\alpha(x) = O(m^{((\alpha/2)-(1/4))})\, e^{x/2} x^{(-(\alpha/2)-(1/12))}, \qquad m \to \infty,$$

uniformly in x, see [4], p. 241. Therefore for all $y \geqslant b$ and fixed $x < b$ we have

$$\begin{aligned} K_m^\alpha(x, y) &= O(m^{1-\alpha})(m^{((\alpha/2)-(1/4))} m^{((\alpha-1)/2)-(1/4)} e^{y/2} y^{(-(\alpha-1)/2)-(1/12)} \\ &\quad + m^{((\alpha-1)/2)-(1/4)} m^{((\alpha/2)-(1/4))} e^{y/2} y^{(-(\alpha/2))-(1/12)}) \\ &= O(1)\, e^{y/2}(y^{(-(\alpha/2))+(5/12)} + y^{(-(\alpha/2))-(1/12)}), \end{aligned}$$

so that

$$\begin{aligned} |I_{m,3}| \leqslant \int_b^\infty |f(y)\, K_m^\alpha(x, y)|\, e^{-y} y^\alpha\, dy &= O(1) \\ &\quad \times \int_b^\infty |f(y)|\, e^{-y/2} y^{((\alpha/2)+(5/12))}\, dy. \end{aligned}$$

Therefore, $I_{m,3}$ can also be made arbitrarily small independently of m by taking b sufficiently large.

Finally we consider $I_{m,2}$. Let $\phi(y)$ be a locally integrable function. Then by Fejer's formula

$$\begin{aligned} &\int_a^b \phi(y)\, L_m^\alpha(y)\, e^{-y} y^\alpha\, dy \\ &\quad = \pi^{-1/2} m^{((\alpha/2)-(1/4))} \left(\int_a^b \phi(y) \cos\left\{2(my)^{1/2} - \frac{2\alpha+1}{4}\pi\right\} \right. \\ &\quad\quad \left. \times e^{-y/2} y^{((\alpha/2)-(1/4))}\, dy + \frac{1}{m^{1/2}} \int_a^b \phi(y)\, \theta_{m,\alpha}(y)\, e^{-y/2} y^{((\alpha/2)-(1/4))} \right) dy \\ &\quad = o(m^{((\alpha/2)-(1/4))}), \qquad m \to \infty. \end{aligned}$$

Therefore, with

$$\phi(y)=\frac{f(y)-f(x)}{x-y},$$

we have

$$I_{m,2}=\int_a^b (f(y)-f(x))\, K_m^\alpha(x,y)\, e^{-y}y^\alpha\, dy$$

$$=O(m^{1-\alpha})\int_a^b \phi(y)(L_{m+1}^\alpha(x)\, L_{m+1}^{\alpha-1}(y)-L_{m+1}^\alpha(y)\, L_{m+1}^{\alpha-1}(x))\, e^{-y}y^\alpha\, dy$$

$$=O(m^{1-\alpha})\left(m^{((\alpha/2)-(1/4))}\int_a^b \phi(y)\, L_{m+1}^{\alpha-1}(y)\, e^{-y}y^\alpha\, dy\right)$$

$$+m^{((\alpha-1)/2)-(1/4)}\int_a^b \phi(y)\, L_{m+1}^\alpha(y)\, e^{-y}y^\alpha\, dy$$

$$=o(1),\qquad m\to\infty.$$

This completes the proof of the theorem.

COROLLARY 3.6. *Let $\alpha\leqslant -1/2$. Suppose there exists a polynomial $P(x)$ such that*

$$\int_0^1 |f(y)-P(y)|\, y^\alpha\, dy<\infty \tag{3.15}$$

and suppose also that

$$\int_1^\infty |f(y)|\, e^{-y/2}y^{((\alpha/2)+(5/12))}\, dy<\infty. \tag{3.16}$$

Furthermore, suppose that $x\in(0,\infty)$ is a point for which

$$\frac{f(y)-f(x)}{y-x} \tag{3.17}$$

is locally integrable on $(0,\infty)$ as a function of y. Then f has an expansion in Laguerre polynomials converging at x to $f(x)$.

Proof. Suppose $P(x)$ is of degree N. There exist constants b_j, $j=0, 1, ..., N$ so that for all $x\in R$:

$$P(x)=\sum_{j=0}^N b_jL_j^\alpha(x).$$

Let a_j be the Laguerre coefficients of $f(x)-P(x)$ defined by

$$a_j=\frac{j!}{\Gamma(j+\alpha+1)}\int_0^\infty (f(y)-P(y))\,L_j^\alpha(y)\,e^{-y}y^\alpha\,dy.$$

We have

$$\sum_{j=0}^{m} a_j L_j^\alpha(x)=\int_0^\infty (f(y)-P(y))\,K_m^\alpha(x,y)\,e^{-y}y^\alpha\,dy.$$

By Theorem 3.5,

$$\lim_{m\to\infty}\int_0^\infty (f(y)-P(y))\,K_m^\alpha(x,y)\,e^{-y}y^\alpha\,dy=f(x)-P(x).$$

Thus

$$\lim_{m\to\infty}\left(\sum_{j=0}^{N} b_j L_j^\alpha(x)+\sum_{j=0}^{m} a_j L_j^\alpha(x)\right)=f(x).$$

THEOREM 3.7. *Let* $-(n+2)<\alpha<-(n+1)$, $n=0, 1, \ldots$ *Suppose there exists a polynomial* $P_n(x)$ *of degree* n *such that*

$$\int_0^1 |f(y)-P_n(y)|\,y^\alpha\,dy<\infty, \tag{3.18}$$

and suppose also that

$$\int_1^\infty |f(y)|\,e^{-y/2}y^{((\alpha/2)+(5/12))}\,dy<\infty. \tag{3.19}$$

Then, if $x\in(0,\infty)$ *is a point for which*

$$\frac{f(y)-f(x)}{y-x} \tag{3.20}$$

is locally integrable on $(0,\infty)$ *as a function of* y *we have*

$$\lim_{m\to\infty} f.p.\int_0^\infty f(y)\,K_m^\alpha(x,y)\,e^{-y}y^\alpha\,dy=f(x). \tag{3.21}$$

Proof. Let $g(x)=f(x)-P_n(x)$. By Theorem 3.5,

$$\lim_{m\to\infty}\int_0^\infty g(y)\,K_m^\alpha(x,y)\,e^{-y}y^\alpha\,dy=g(x).$$

On the other hand, for all $m \geqslant n$ we have

$$P_n(x) = f.p. \int_0^\infty P_n(y)\, K_m^\alpha(x, y)\, e^{-y} y^\alpha\, dy.$$

Therefore

$$\begin{aligned}
&\lim_{m \to \infty} f.p. \int_0^\infty f(y)\, K_m^\alpha(x, y)\, e^{-y} y^\alpha\, dy \\
&\quad = \lim_{m \to \infty} \left(\int_0^\infty (f(y) - P_n(y))\, K_m^\alpha(x, y)\, e^{-y} y^\alpha\, dy \right. \\
&\quad \left. + f.p. \int_0^\infty P_n(y)\, K_m^\alpha(x, y)\, e^{-y} y^\alpha\, dy \right) \\
&\quad = f(x).
\end{aligned}$$

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